Problem 2.27

A conical surface (an empty ice-cream cone) carries a uniform surface charge σ. The height of the cone is h, as is the radius of the top. Find the potential difference between points **a** (the vertex) and **b** (the center of the top).

Solution

The two equations governing the electric (electrostatic) field are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla \times \mathbf{E} = \mathbf{0}. \tag{2}$$

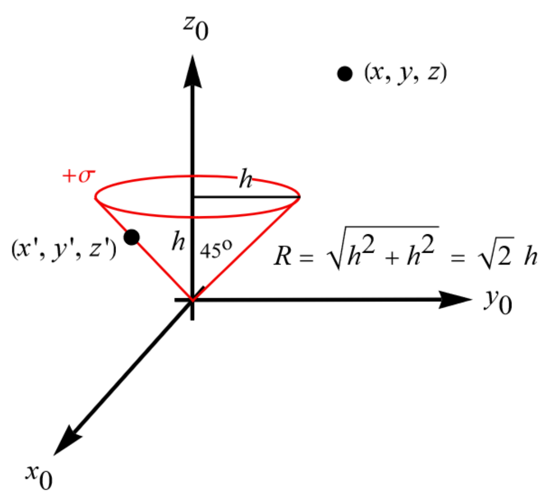
Equation (2) implies the existence of a potential function $-V$ that satisfies

$$\mathbf{E} = \nabla(-V) = -\nabla V. \tag{3}$$

The minus sign is arbitrary mathematically, but physically it indicates that a positive charge in an electric field moves from high-potential regions to low-potential regions (and vice-versa for a negative charge). Since we want the potential difference between **a** and **b**, integrate both sides of equation (3) along a path from **a** to **b** and use the fundamental theorem for line integrals on the right side.

$$\begin{aligned}\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}_0 &= -\int_{\mathbf{a}}^{\mathbf{b}} \nabla V \cdot d\mathbf{l}_0 \\ &= -[V(\mathbf{b}) - V(\mathbf{a})] \\ &= V(\mathbf{a}) - V(\mathbf{b})\end{aligned}$$

Here **a** is the vector to the vertex of the inverted cone illustrated below, and **b** is the vector to the center of the base: $\mathbf{a} = \langle 0, 0, 0\rangle$ and $\mathbf{b} = \langle 0, 0, h\rangle$.



(x, y, z) is the point in space we want to know the electric potential or electric field, and (x', y', z') is a point on the charge distribution.

$$V(\mathbf{a}) - V(\mathbf{b}) = \int_{\langle 0,0,0\rangle}^{\langle 0,0,h\rangle} \mathbf{E} \cdot d\mathbf{l}_0$$

Method I: Finding the Potential Difference by Evaluating the Right Side

Parameterize the integration path from $\langle 0, 0, 0\rangle$ to $\langle 0, 0, h\rangle$ as follows.

$$x_0 = 0, \quad y_0 = 0, \quad z_0 = t_0, \quad 0 \le t_0 \le h$$

Consequently,

$$\begin{aligned} V(\mathbf{a}) - V(\mathbf{b}) &= \int_0^h \mathbf{E}(\mathbf{l}_0(t_0)) \cdot \mathbf{l}_0'(t_0)\, dt_0 \\ &= \int_0^h \mathbf{E}(0, 0, t_0) \cdot \langle 0, 0, 1\rangle\, dt_0. \end{aligned} \tag{4}$$

The aim now is to calculate the electric field at the point $(0, 0, t_0)$. Start with the basic formula for a surface charge distribution,

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \iint_{\mathcal{S}} \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, da' \\ &= \frac{1}{4\pi\epsilon_0} \iint_{\mathcal{S}} \frac{\sigma}{|\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\, da' \\ &= \frac{\sigma}{4\pi\epsilon_0} \iint_{\mathcal{S}} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\, da'. \end{aligned}$$

For the point $(0, 0, t_0)$ in particular,

$$\begin{aligned} \mathbf{E}(0, 0, t_0) &= \frac{\sigma}{4\pi\epsilon_0} \iint_{\mathcal{S}} \frac{\langle 0, 0, t_0\rangle - \langle x', y', z'\rangle}{|\mathbf{r} - \mathbf{r}'|^3}\, da' \\ &= \frac{\sigma}{4\pi\epsilon_0} \iint_{\mathcal{S}} \frac{\langle -x', -y', t_0 - z'\rangle}{\left[\sqrt{(-x')^2 + (-y')^2 + (t_0 - z')^2}\right]^3}\, da' \\ &= -\frac{\sigma}{4\pi\epsilon_0} \iint_{\mathcal{S}} \frac{\langle x', y', z' - t_0\rangle}{[x'^2 + y'^2 + (z' - t_0)^2]^{3/2}}\, da'. \end{aligned}$$

Since $\mathcal{S}$ is the inverted cone here, switch to spherical coordinates (r, ϕ, θ), where θ is the angle from the polar axis.

$$\begin{aligned} x' &= r' \sin 45^\circ \cos\phi' \\ y' &= r' \sin 45^\circ \sin\phi' \\ z' &= r' \cos 45^\circ \end{aligned}$$

Note that for a cone, $da = r\sin\theta\,dr\,d\phi$.

$$\begin{aligned}
\mathbf{E}(0,0,t_0) &= -\frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\sqrt{2}h}\frac{\langle r'\sin 45^\circ\cos\phi', r'\sin 45^\circ\sin\phi', r'\cos 45^\circ - t_0\rangle}{[(r'\sin 45^\circ\cos\phi')^2 + (r'\sin 45^\circ\sin\phi')^2 + (r'\cos 45^\circ - t_0)^2]^{3/2}}\,(r'\sin 45^\circ\,dr'\,d\phi'). \\
&= -\frac{\sigma\sin 45^\circ}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\sqrt{2}h}\frac{r'}{\left(r'^2 - 2r't_0\cos 45^\circ + t_0^2\right)^{3/2}}\langle r'\sin 45^\circ\cos\phi', r'\sin 45^\circ\sin\phi', r'\cos 45^\circ - t_0\rangle\,dr'\,d\phi' \\
&= -\frac{\sigma\sin 45^\circ}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\frac{r'}{\left(r'^2 - 2r't_0\cos 45^\circ + t_0^2\right)^{3/2}}\left\langle r'\sin 45^\circ\int_0^{2\pi}\cos\phi'\,d\phi',\right. \\
&\qquad\qquad r'\sin 45^\circ\int_0^{2\pi}\sin\phi'\,d\phi', \\
&\qquad\qquad \left.(r'\cos 45^\circ - t_0)\int_0^{2\pi}d\phi'\right\rangle\,dr' \\
&= -\frac{\sigma\sin 45^\circ}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\frac{r'}{\left(r'^2 - 2r't_0\cos 45^\circ + t_0^2\right)^{3/2}}\left\langle r'\sin 45^\circ(0), r'\sin 45^\circ(0), (r'\cos 45^\circ - t_0)(2\pi)\right\rangle\,dr' \\
&= -\frac{\sigma\sin 45^\circ}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\frac{r'}{\left(r'^2 - 2r't_0\cos 45^\circ + t_0^2\right)^{3/2}}\left\langle 0, 0, 2\pi(r'\cos 45^\circ - t_0)\right\rangle\,dr' \\
&= -\frac{\sigma\sin 45^\circ}{2\epsilon_0}\langle 0,0,1\rangle\int_0^{\sqrt{2}h}\frac{r'(r'\cos 45^\circ - t_0)}{\left(r'^2 - 2r't_0\cos 45^\circ + t_0^2\right)^{3/2}}\,dr' \\
&= \frac{\sigma\sin 45^\circ}{2\epsilon_0}\langle 0,0,1\rangle\int_0^{\sqrt{2}h}\frac{r'(t_0 - r'\cos 45^\circ)}{\left(t_0^2 - 2r't_0\cos 45^\circ + r'^2\right)^{3/2}}\,dr'
\end{aligned}$$

Plug this result into equation (4).

$$\begin{aligned}
V(\mathbf{a}) - V(\mathbf{b}) &= \int_0^h \mathbf{E}(0,0,t_0)\cdot\langle 0,0,1\rangle\,dt_0 \qquad (4) \\
&= \frac{\sigma\sin 45^\circ}{2\epsilon_0}(\langle 0,0,1\rangle\cdot\langle 0,0,1\rangle)\int_0^h\int_0^{\sqrt{2}h}\frac{r'(t_0 - r'\cos 45^\circ)}{\left(t_0^2 - 2r't_0\cos 45^\circ + r'^2\right)^{3/2}}\,dr'\,dt_0 \\
&= \frac{\sigma\sin 45^\circ}{2\epsilon_0}(1)\int_0^{\sqrt{2}h} r'\left[\int_0^h\frac{t_0 - r'\cos 45^\circ}{\left(t_0^2 - 2r't_0\cos 45^\circ + r'^2\right)^{3/2}}\,dt_0\right]dr'
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
u &= t_0^2 - 2r't_0\cos 45^\circ + r'^2 \\
du &= (2t_0 - 2r'\cos 45^\circ)\,dt_0 \quad\rightarrow\quad \frac{du}{2} = (t_0 - r'\cos 45^\circ)\,dt_0
\end{aligned}$$

So then

$$
\begin{aligned}
V(\mathbf{a}) - V(\mathbf{b}) &= \frac{\sigma \sin 45^\circ}{2\epsilon_0} \int_0^{\sqrt{2}h} r' \left[\int_{r'^2}^{h^2 - 2r'h\cos 45^\circ + r'^2} \frac{1}{u^{3/2}} \left(\frac{du}{2} \right) \right] dr' \\
&= \frac{\sigma \sin 45^\circ}{4\epsilon_0} \int_0^{\sqrt{2}h} r' \left(\int_{r'^2}^{h^2 - 2r'h\cos 45^\circ + r'^2} u^{-3/2}\, du \right) dr' \\
&= \frac{\sigma \sin 45^\circ}{4\epsilon_0} \int_0^{\sqrt{2}h} r' \left(\frac{-2}{\sqrt{u}} \right) \Bigg|_{r'^2}^{h^2 - 2r'h\cos 45^\circ + r'^2} dr' \\
&= \frac{\sigma \sin 45^\circ}{2\epsilon_0} \int_0^{\sqrt{2}h} r' \left(\frac{1}{r'} - \frac{1}{\sqrt{h^2 - 2r'h\cos 45^\circ + r'^2}} \right) dr' \\
&= \frac{\sigma \sin 45^\circ}{2\epsilon_0} \left(\int_0^{\sqrt{2}h} dr' - \int_0^{\sqrt{2}h} \frac{r'}{\sqrt{r'^2 - 2r'h\cos 45^\circ + h^2}}\, dr' \right) \\
&= \frac{\sigma}{2\sqrt{2}\epsilon_0} \left(\sqrt{2}h - \int_0^{\sqrt{2}h} \frac{r'}{\sqrt{r'^2 - \frac{2}{\sqrt{2}}hr' + h^2}}\, dr' \right) \\
&= \frac{\sigma}{2\sqrt{2}\epsilon_0} \left[\sqrt{2}h - \int_0^{\sqrt{2}h} \frac{r'}{h\sqrt{\left(\frac{r'}{h}\right)^2 - \frac{2}{\sqrt{2}}\left(\frac{r'}{h}\right) + 1}}\, dr' \right].
\end{aligned}
$$

Make another substitution.

$$
\begin{aligned}
v &= \frac{r'}{h} \\
dv &= \frac{dr'}{h} \quad \rightarrow \quad h\, dv = dr'
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
V(\mathbf{a}) - V(\mathbf{b}) &= \frac{\sigma}{2\sqrt{2}\epsilon_0} \left[\sqrt{2}h - \int_0^{\sqrt{2}} \frac{v}{\sqrt{v^2 - \frac{2}{\sqrt{2}}v + 1}} (h\, dv) \right] \\
&= \frac{\sigma h}{2\sqrt{2}\epsilon_0} \left[\sqrt{2} - \int_0^{\sqrt{2}} \frac{\left(v - \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}}{\sqrt{v^2 - \frac{2}{\sqrt{2}}v + 1}}\, dv \right] \\
&= \frac{\sigma h}{2\sqrt{2}\epsilon_0} \left[\sqrt{2} - \left(\int_0^{\sqrt{2}} \frac{v - \frac{1}{\sqrt{2}}}{\sqrt{v^2 - \frac{2}{\sqrt{2}}v + 1}}\, dv + \frac{1}{\sqrt{2}} \int_0^{\sqrt{2}} \frac{dv}{\sqrt{v^2 - \frac{2}{\sqrt{2}}v + 1}} \right) \right].
\end{aligned}
$$

Make a substitution in the first integral.

$$
\begin{aligned}
w &= v^2 - \frac{2}{\sqrt{2}}v + 1 \\
dw &= \left(2v - \frac{2}{\sqrt{2}}\right) dv \quad \rightarrow \quad \frac{dw}{2} = \left(v - \frac{1}{\sqrt{2}}\right) dv
\end{aligned}
$$

As a result, the limits in the first integral become the same. Complete the square in the denominator of the second integrand.

$$\begin{aligned} V(\mathbf{a}) - V(\mathbf{b}) &= \frac{\sigma h}{2\sqrt{2}\epsilon_0}\left[\sqrt{2} - \left(\underbrace{\int_1^1 \frac{1}{\sqrt{w}}\frac{dw}{2}}_{=0} + \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} \frac{dv}{\sqrt{v^2 - \frac{2}{\sqrt{2}}v + 1}}\right)\right] \\ &= \frac{\sigma h}{2\sqrt{2}\epsilon_0}\left(\sqrt{2} - \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} \frac{dv}{\sqrt{v^2 - \sqrt{2}v + 1}}\right) \\ &= \frac{\sigma h}{2\epsilon_0}\left(1 - \frac{1}{2}\int_0^{\sqrt{2}} \frac{dv}{\sqrt{v^2 - \sqrt{2}v + \frac{1}{2} - \frac{1}{2} + 1}}\right) \\ &= \frac{\sigma h}{2\epsilon_0}\left[1 - \frac{1}{2}\int_0^{\sqrt{2}} \frac{dv}{\sqrt{\left(v - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}}\right] \end{aligned}$$

Make the following substitution.

$$\begin{aligned} v - \frac{1}{\sqrt{2}} &= \frac{1}{\sqrt{2}}\tan\alpha \quad \rightarrow \quad \left(v - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} = \frac{1}{2}\sec^2\alpha \\ dv &= \frac{1}{\sqrt{2}}\sec^2\alpha\,d\alpha \end{aligned}$$

So then

$$\begin{aligned} V(\mathbf{a}) - V(\mathbf{b}) &= \frac{\sigma h}{2\epsilon_0}\left(1 - \frac{1}{2}\int_{\tan^{-1}(-1)}^{\tan^{-1}(1)} \frac{\frac{1}{\sqrt{2}}\sec^2\alpha\,d\alpha}{\sqrt{\frac{1}{2}\sec^2\alpha}}\right) \\ &= \frac{\sigma h}{2\epsilon_0}\left(1 - \frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec\alpha\,d\alpha\right) \\ &= \frac{\sigma h}{2\epsilon_0}\left(1 - \frac{1}{2} \times 2\int_0^{\pi/4} \sec\alpha\,d\alpha\right) \\ &= \frac{\sigma h}{2\epsilon_0}\left(1 - \ln|\sec\alpha + \tan\alpha|\Big|_0^{\pi/4}\right) \\ &= \frac{\sigma h}{2\epsilon_0}\left[1 - \ln(\sqrt{2} + 1) + \ln(1)\right]. \end{aligned}$$

Therefore, the potential difference between the vertex and the center of the top is

$$V(\mathbf{a}) - V(\mathbf{b}) = \frac{\sigma h}{2\epsilon_0}\left[1 - \ln\left(\sqrt{2} + 1\right)\right].$$

Method II: Finding the Potential Difference by Evaluating the Left Side

Alternatively, the potential difference can be obtained by evaluating $V(\mathbf{a}) = V(0,0,0)$ and $V(\mathbf{b}) = V(0,0,h)$ directly and taking the difference. Begin with the basic formula for the electric potential of a continuous surface charge distribution.

$$\begin{aligned}
V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \iint_{\mathcal{S}} \frac{\sigma(\mathbf{r}')}{\mathscr{r}}\, da' \\
&= \frac{1}{4\pi\epsilon_0} \iint_{\mathcal{S}} \frac{\sigma}{|\mathbf{r}-\mathbf{r}'|}\, da' \\
&= \frac{\sigma}{4\pi\epsilon_0} \iint_{\mathcal{S}} \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\, da' \\
&= \frac{\sigma}{4\pi\epsilon_0} \iint_{\mathcal{S}} \frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}\, da'
\end{aligned}$$

Since $\mathcal{S}$ is the inverted cone here, switch to spherical coordinates (r, ϕ, θ), where θ is the angle from the polar axis.

$$\begin{aligned}
x' &= r' \sin 45^\circ \cos\phi' \\
y' &= r' \sin 45^\circ \sin\phi' \\
z' &= r' \cos 45^\circ
\end{aligned}$$

Note that for a cone, $da = r \sin\theta \, dr \, d\phi$.

$$\begin{aligned}
V(\mathbf{r}) &= \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\sqrt{2}h} \frac{1}{\sqrt{(r'\sin 45^\circ \cos\phi' - x)^2 + (r'\sin 45^\circ \sin\phi' - y)^2 + (r'\cos 45^\circ - z)^2}} (r'\sin 45^\circ\, dr'\, d\phi') \\
&= \frac{\sigma \sin 45^\circ}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\sqrt{2}h} \frac{r'}{\sqrt{r'^2 - 2r'(x\sin 45^\circ \cos\phi' + y \sin 45^\circ \sin\phi' + z\cos 45^\circ) + x^2 + y^2 + z^2}}\, dr'\, d\phi'
\end{aligned}$$

Calculate $V(\mathbf{a}) = V(0,0,0)$ first.

$$\begin{aligned}
V(\mathbf{a}) &= \frac{\sigma \sin 45^\circ}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\sqrt{2}h} \frac{r'}{\sqrt{r'^2}}\, dr'\, d\phi' \\
&= \frac{\sigma \sin 45^\circ}{4\pi\epsilon_0} \left(\int_0^{2\pi} d\phi' \right) \left(\int_0^{\sqrt{2}h} dr' \right) \\
&= \frac{\sigma}{4\sqrt{2}\pi\epsilon_0} (2\pi)(\sqrt{2}h) \\
&= \frac{\sigma h}{2\epsilon_0}
\end{aligned}$$

Now calculate $V(\mathbf{b}) = V(0,0,h)$.

$$\begin{aligned}
V(\mathbf{b}) &= \frac{\sigma \sin 45^\circ}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\sqrt{2}h} \frac{r'}{\sqrt{r'^2 - 2r'(h\cos 45^\circ) + h^2}}\, dr'\, d\phi' \\
&= \frac{\sigma \sin 45^\circ}{4\pi\epsilon_0} \left(\int_0^{2\pi} d\phi' \right) \left(\int_0^{\sqrt{2}h} \frac{r'}{\sqrt{r'^2 - \frac{2}{\sqrt{2}} r' h + h^2}}\, dr' \right) \\
&= \frac{\sigma}{4\sqrt{2}\pi\epsilon_0} (2\pi) \left[h\sqrt{2} \ln\left(\sqrt{2} + 1\right) \right] \\
&= \frac{\sigma h}{2\epsilon_0} \ln\left(\sqrt{2} + 1\right)
\end{aligned}$$

Therefore, the potential difference between the vertex and the center of the top is

$$\begin{aligned}
V(\mathbf{a}) - V(\mathbf{b}) &= \frac{\sigma h}{2\epsilon_0} - \frac{\sigma h}{2\epsilon_0} \ln\left(\sqrt{2} + 1\right) \\
&= \frac{\sigma h}{2\epsilon_0} \left[1 - \ln\left(\sqrt{2} + 1\right) \right].
\end{aligned}$$